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The Energy Level Spacing for Two Harmonic Oscillators with Generic Ratio of Frequencies

P. M. Bleher[1]

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The limit distribution of energy level spacing is studied for the system of two harmonic oscillators with generic ratio of frequencies. It is proved that for any fixed generic ratio α no limit distribution exists, but for random α with any absolutely continuous distribution $p(\alpha)\,d\alpha$ on $[0, 1]$ a universal random limit distribution of the energy level spacing exists. Some properties of the random limit distribution are discussed.

KEY WORDS: Two-dimensional quantum harmonic oscillator; distribution of energy level spacing; continued fractions; the Gauss map; natural extension of invariant measures; universality and rigidity of the spectrum.

1. INTRODUCTION

This work derives from the attempts to prove or to disprove the following general conjecture discussed in ref. 1: Energy level spacing in the spectral interval $E_0 < E < E_1$ for two harmonic oscillators with generic ratio of frequencies has a limit distribution when $E_1 \to \infty$. It turns out that the situation is a little bit unusual and in the present paper we *prove* and *disprove* this conjecture simultaneously. The point is that for any *fixed* generic irrational ratio α of frequencies no limit distribution exists, but for *random* α with any absolutely continuous distribution on $[0, 1]$ a *random* limit distribution of energy level spacing *exists.*

The Hamiltonian of the model is

$$H = \frac{p_1^2 + \omega_1^2 q_1^2}{2} + \frac{p_2^2 + \omega_2^2 q_2^2}{2}, \qquad \omega_1, \omega_2 > 0$$

[1] Department of Mathematics, Hill Center, Busch Campus, Rutgers University, New Brunswick, New Jersey 08903, and School of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv, 69978, Israel.

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The quantum energy levels of the system are labeled by two integer numbers $m, n \geqslant 0$ and they have the form

$$E_{mn} = E_{00} + m\omega_1 + n\omega_2$$

The problem is to study the distribution of energy level spacing (the distance between neighbor levels) in the spectral integral $E_{00} \leqslant E_{mn} \leqslant E$ when $E \to \infty$. Since

$$E_{mn} = E_{00} + \omega_1(m + n\alpha)$$

where $\alpha = \omega_2/\omega_1 > 0$, the problem is reduced to the similar one for the sequence

$$\lambda_{mn} = m + n\alpha, \qquad m, n \geqslant 0$$

The case of rational α is not interesting, because of the strong degeneracy of $\{\lambda_{mn}\}$ (see ref. 1); hence we shall assume that α is irrational. Without loss of generality we may assume that $0 < \alpha < 1$.

The distribution of neighbor distances in the sequence $\lambda_{mn} = m + n\alpha$ for the first time was considered in ref. 1. In that paper is was shown that for generic α there is no energy level clustering, which was observed for nonlinear integrable systems, and some other properties of the spacing distribution were studied both theoretically and numerically.

In ref. 2 it was discovered that locally the distance between neighbor energy levels can take only three values, which means *strong rigidity* of the local structure of the spectrum and implies strong "repulsion of energy levels." It was shown also that the local spacing distribution is fluctuating, so that no limit of this distribution exists when the spectral interval goes to ∞.

In ref. 3 a particular case of the golden mean $\alpha = (\sqrt{5} - 1)/2$ was considered. It was proved that in that case the limit spacing distribution does exist if the limit is taken by the spectral intervals $\{0 \leqslant \lambda_{mn} < f_j\}$, $j \to \infty$, where $f_1, f_2, f_3,...$ are the Fibonacci numbers, and it does not exist for general sequence $\{0 \leqslant \lambda_{mn} < \lambda\}$, $\Lambda \to \infty$.

In the present paper we develop the approach suggested in ref. 3 to study the case of generic α. Let

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

$\alpha = [a_1, a_2,...]$, be the expansion of α into the continued fraction, and

$$\frac{p_j}{q_j} = [a_1, a_2,..., a_j], \qquad j \geqslant 1$$

be the approximants of α. Denote

$$\Omega_j = \{\lambda_{mn} = m + n\alpha \mid m, n \geqslant 0;\ p_j > \lambda_{mn} \geqslant 0\}$$

and let $0 < \varepsilon_{j0} < \varepsilon_{j1} < \varepsilon_{j2} < \cdots$ be all the *different* neighbor distances in Ω_j. As we shall see, each such distance ε_{jl} is *multiple*, i.e., there exist many pairs of neighbor elements λ_{mn}, $\lambda_{m'n'} \in \Omega_j$ with $|\lambda_{mn} - \lambda_{m'n'}| = \varepsilon_{jl}$. Denote by $L_{jl} \geqslant 1$ the *multiplicity* of the distance ε_{jl} and put

$$\pi_{jl} = \frac{L_{jl}}{|\Omega_j| - 1}$$

where $|\Omega_j|$ is the number of elements in Ω_j. It is clear, that π_{jl} is the *fraction* of ε_{jl} among all the neighbor distances, or the *probability* of ε_{jl} with respect to the uniform distribution on the set of neighbor distances. Denote by

$$s_{jl} = \frac{\varepsilon_{jl}}{\varepsilon_{j0}}$$

the normalized to ε_{j0} neighbor distances ε_{jl} and by

$$\rho_j(ds) = \sum_l \pi_{jl}\, \delta(s - s_{jl})\, ds$$

the distribution of the normalized neighbor distances. Remark that s_{jl}, π_{jl}, and $\rho_j(ds)$ depend on α. The main problem we are interested in is the existence of a limit of the sequence of the distributions $\rho_j(ds)$ when $j \to \infty$.

We will use weak convergence of probability measures and random variables (see ref. 4). Recall that a sequence of probability measures $\mu_n(dx)$ in $\mathbf{R}^k$ converges weakly to a probability measure $\mu(dx)$ iff for any bounded continuous function $f(x)$,

$$\lim_{n \to \infty} \int_{\mathbf{R}^k} f(x)\, \mu_n(dx) = \int_{\mathbf{R}^k} f(x)\, \mu(dx) \tag{1.1}$$

Respectively, a sequence of random variables $\xi_n \in \mathbf{R}^k$ converges weakly to a random variable $\xi \in \mathbf{R}^k$ iff the probability distributions of ξ_n converge weakly to the one of ξ. We will write in such cases that

$$\mu = w\text{-}\lim_{n \to \infty} \mu_n, \qquad \xi = w\text{-}\lim_{n \to \infty} \xi_n$$

Theorem 1.1. Let α be a random variable on $[0, 1]$ with an absolutely continuous distribution $p(\alpha)\, d\alpha$. Then for any $l \geqslant 0$ there exist

$w\text{-}\lim_{j\to\infty} s_{jl} = s_l$ and $w\text{-}\lim_{j\to\infty} \pi_{jl} = \pi_l$ and the limit random variables s_l and π_l do not depend on the distribution $p(\alpha)\,d\alpha$.

The following theorem gives uniform estimates for s_{jl} and π_{jl}.

Theorem 1.2. For any α we have the estimate $2^l \geqslant s_{jl} \geqslant l/2$ and $\pi_{jl} \leqslant C/(1+l)^2$, where C is an absolute constant.

Next we describe the joint distribution of the random variables $\{s_l = w\text{-}\lim_{j\to\infty} s_{jl},\ \pi_l = w\text{-}\lim_{j\to\infty} \mu_{jl};\ l \geqslant 0\}$. To do this we need to introduce some notations. Let

$$G:\quad \alpha \to \left\{\frac{1}{\alpha}\right\}$$

be the Gauss map and

$$\mu(d\alpha) = \frac{d\alpha}{(1+\alpha)\ln 2}$$

be the absolutely continuous invariant measure of this map. Let $\mu_\infty(d\alpha\, d\beta)$ be the natural extension of $\mu(d\alpha)$. It is a probability measure on the unit square $[0,1]\times[0,1]$ and it is invariant and ergodic with respect to the map

$$G_\infty:\quad (\alpha, \beta) \to \left(\left\{\frac{1}{\alpha}\right\}, \frac{1}{[1/\alpha]+\beta}\right)$$

(see Section 3 below). It is worth to note that $\mu_\infty(d\alpha\, d\beta)$ is singularly continuous with respect to Lebesque measure $d\alpha\, d\beta$ and its support coincides with the unit square $[0,1]\times[0,1]$.

Let $\gamma = (\alpha, \beta) \in [0,1]\times[0,1]$ be a random variable with the distribution $\mu_\infty(d\alpha\, d\beta)$.

Theorem 1.3. Under the assumptions of Theorem 1.1 there exist bounded functions $F_l(\gamma)$, $R_l(\gamma)$ on $[0,1]\times[0,1]$, which are continuous at almost all γ (with respect to the distribution μ_∞ of γ), such that $s_l = F_l(\gamma)$, $\pi_l = R_l(\gamma)$, $l \geqslant 0$. The functions F_l, R_l, $l \geqslant 0$, are universal in the sense that they do not depend on the distribution $p(\alpha)\,d\alpha$.

Theorems 1.2 and 1.3 enable us to prove also the following statement.

Theorem 1.4. Under the assumptions of Theorem 1.1

$$\lim_{j\to\infty} \rho_j(ds) = \rho(ds) = \sum_{l=0}^{\infty} \pi_l \delta(s - s_l)\, ds$$

The limit is understood as w-lim of the distribution functions $P_j(x) = \int_{-\infty}^{x} \rho_j(ds) = \sum_{\{l \mid s_{jl} \leqslant x\}} \pi_{jl}$:

$$w\text{-}\lim_{j \to \infty} P_j(x) = P(x) = \int_{-\infty}^{x} \rho(ds) = \sum_{\{l \mid s_l \leqslant x\}} \pi_l$$

for any $x \geqslant 0$.

Our last theorem shows that for generic *fixed* α the sequence $\rho_j(ds)$ has no limit when $j \to \infty$.

Theorem 1.5. For almost all α the sequence π_{j0} has no limit when $j \to \infty$.

Remark. The same can be shown for any π_{jl}, $l \geqslant 0$, and s_{jl}, $l \geqslant 1$.

2. EXACT FORMULAS FOR NEIGHBOR DISTANCES

Let $0 < \alpha < 1$ be an irrational number. Expand it into the continued fraction, $\alpha = [a_1, a_2, \ldots]$. Recall some definitions and properties of continued fractions (see, e.g., ref. 5). The sign $\equiv$ below means a definition, the sign $=$ means an equality. We have

$$\frac{p_n}{q_n} \equiv [a_1, a_2, \ldots, a_n], \qquad (p_n, q_n) = 1$$

$$p_n = a_n p_{n-1} + p_{n-2}; \qquad p_{-1} \equiv 1, \qquad p_0 \equiv 0, \quad p_1 = 1$$

$$q_n = a_n q_{n-1} + q_{n-2}; \qquad q_{-1} \equiv 0, \quad q_0 \equiv 1, \quad q_1 = a_1$$

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n$$

$$\frac{q_{n-1}}{q_n} = [a_n, a_{n-1}, \ldots, a_1]$$

$$\varepsilon_n \equiv |q_n \alpha - p_n| = (-1)^n (q_n \alpha - p_n)$$

$$\varepsilon_n = -a_n \varepsilon_{n-1} + \varepsilon_{n-2}; \qquad \varepsilon_{-1} = 1, \quad \varepsilon_0 = \alpha$$

$$q_n \varepsilon_{n-1} + q_{n-1} \varepsilon_n = 1$$

$$r_n \equiv \frac{\varepsilon_n}{\varepsilon_{n-1}}; \qquad r_{n-1} = \frac{1}{a_n + r_n}$$

$$r_n = G[r_{n-1}] \equiv \left\{ \frac{1}{r_{n-1}} \right\}, \qquad r_0 = \alpha$$

$$a_n = A[r_{n-1}] \equiv \left[\frac{1}{r_{n-1}} \right]$$

$$\varepsilon_n = r_n \varepsilon_{n-1} = \cdots = r_n \cdots r_0 = G^n[\alpha] \cdots G^0[\alpha]$$

$$G^0[\alpha] \equiv \alpha, \qquad G^n[\alpha] \equiv G[G^{n-1}[\alpha]]$$
$$a_n = AG^{n-1}[\alpha] \equiv A[G^{n-1}[\alpha]]$$
$$r_n = G^n[\alpha]$$

All the subsequent considerations are based on the following two propositions.

Proposition 2.1. The set of neighbor distances in the set $M_k = \{k-1 \leqslant m + \alpha n \leqslant k;\ m, n \geqslant 0\}$, $k \in \mathbf{N}$, coincides with the same in the set $N_l = \{\{m\alpha\},\ 0 \leqslant m < l\}$, where $l = [k/\alpha] + 1$, which is considered on the circle $S^1 = [0, 1]$, $0 = 1$.

Proposition 2.2. Let $1 \leqslant i \leqslant a_j$. Then if $q_{j-2} + iq_{j-1} < l \leqslant q_{j-2} + (i+1)q_{j-1}$, then the neighbor distances in the set N_l can be only one of the following three numbers: ε_{j-1}, $\varepsilon_{j-2} - (i-1)\varepsilon_{j-1}$, $\varepsilon_{j-2} - i\varepsilon_{j-1}$. The numbers $\lambda(l; \varepsilon)$ of the neighbor distances of the length ε are equal to

$$\begin{aligned} \lambda(l; \varepsilon_{j-1}) &= l - q_{j-1} \\ \lambda(l; \varepsilon_{j-2} - (i-1)\varepsilon_{j-1}) &= q_{j-2} + (i+1)q_{j-1} - l \\ \lambda(l; \varepsilon_{j-2} - i\varepsilon_{j-1}) &= l - q_{j-2} - iq_{j-1} \end{aligned} \tag{2.1}$$

Proposition 2.1 in proved in refs. 2 and 3. Proposition 2.2 is well known and we omit its proof. Remark only that it has a simple visual explanation: When we add the point $\{l_\alpha\}$ to N_l to obtain N_{l+1}, some segment of length $\varepsilon_{j-2} - (i-1)\varepsilon_{j-1}$ is split into two segments of lengths $\varepsilon_{j-2} - i\varepsilon_{j-1}$ and ε_{j-1}. It continues until all the segments of length $\varepsilon_{j-2} - (i-1)\varepsilon_{j-1}$ are exhausted. Then the process begins of splitting the segments of length $\varepsilon_{j-2} - i\varepsilon_{j-1}$ into two segments of lengths $\varepsilon_{j-2} - (i+1)\varepsilon_{j-1}$ and ε_{j-1} and so on.

Remark that $\varepsilon_{j-2} - a_j\varepsilon_{j-1} = \varepsilon_j$, so $\varepsilon_{j-2} - i\varepsilon_{j-1} = \varepsilon_j + (a_j - i)\varepsilon_{j-1} = \varepsilon_j + k\varepsilon_{j-1}$, where $k = a_j - i$. We shall call the sequence $\varepsilon_j + \varepsilon_{j-1}$, $\varepsilon_j + 2\varepsilon_{j-1}, \ldots,$ $\varepsilon_j + a_j\varepsilon_{j-1} = \varepsilon_{j-2}$ the *jth series* of neighbor distances and we shall denote it by E_j. One can see easily that any element from E_j is less than any element from E_{j-1}. Denote $E = \bigcup_{j \geqslant 1} E_j$.

Corollary of Proposition 2.2. All possible neighbor distances in the sets N_l can be only elements from the set $E = \bigcup_{j \geqslant 1} E_j = \bigcup_{j \geqslant 1} \{\varepsilon_j + k\varepsilon_{j-1},\ 1 \leqslant k \leqslant a_j\}$ and the number $\lambda(l; \varepsilon_j + k\varepsilon_{j-1})$ of the neighbor distances of length $\varepsilon_j + k\varepsilon_{j-1}$ in the set N_l is equal to

$$\begin{aligned} \lambda(l; \varepsilon_j + k\varepsilon_{j-1}) &= q_{j-1} - |l - q_{j-2} - (a_j - k + 1)q_{j-1}| \\ &\quad \text{if} \quad q_{j-2} + (a_j - k)q_{j-1} \leqslant l \leqslant q_{j-2} + (a_j - k + 2)q_{j-1} \\ \lambda(l; \varepsilon_j + k\varepsilon_{j-1}) &= 0 \qquad \text{otherwise} \end{aligned} \tag{2.2}$$

Thus $\lambda(l;\varepsilon_j+k\varepsilon_{j-1})$ is maximal at $l=q_{j-2}+(a_j-k+1)q_{j-1}$, where it is equal to q_{j-1}, and it decreases linearly from both sides of the maximum point (see Fig. 1a). Denote

$$M(\Lambda)=\{\lambda_{mn}=m+\alpha n \mid m,n\geqslant 0;\ 0\leqslant\lambda_{mn}<\Lambda\}$$

For $A\subset M(\Lambda)$ put $\Pr A\equiv|A|/|M(\Lambda)|$, where $|A|$ means the number of elements in A, so that $\Pr A$ is the probability of A with respect to the uniform distribution in $M(\Lambda)$. Let $\varepsilon=\varepsilon(\lambda_{mn})$ be a function on $M(\Lambda)$ which corresponds to $\lambda_{mn}\in M(\Lambda)$, the distance $\Delta_{mn}=\lambda_{mn}-\lambda_{m'n'}$ from λ_{mn} to

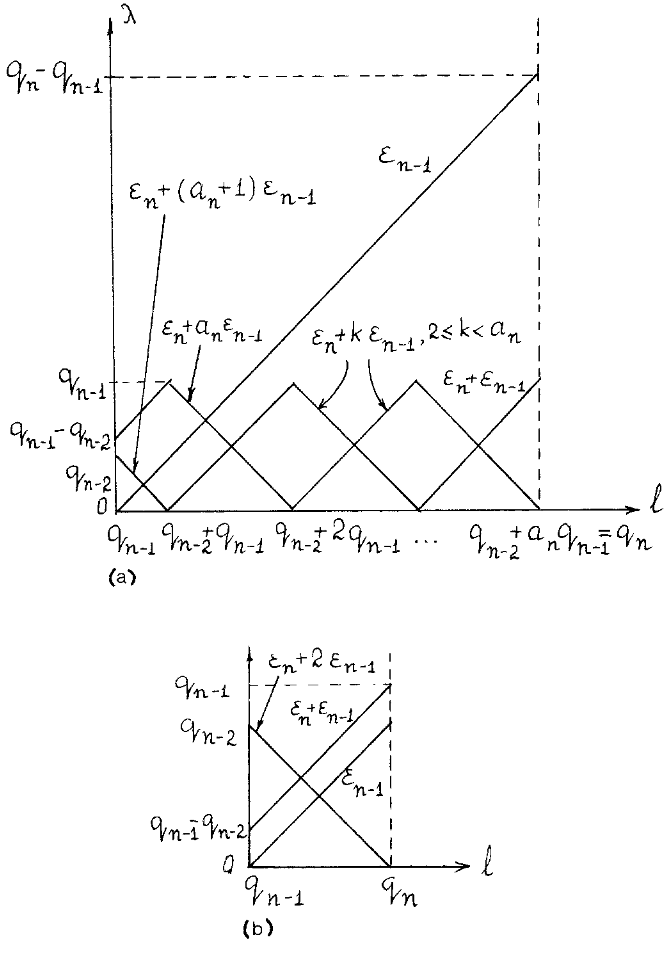

Fig. 1. The graphs of the functions $\lambda(l;\varepsilon)$ on the segment $q_{n-1}\leqslant l\leqslant q_n$ for various ε. (a) $a_n\geqslant 2$. (b) $a_n=1$.

the neighbor $\lambda_{m'n'}$ (ε is not defined for λ_{00}). Denote $A(\varepsilon^0; \Lambda) = \{\lambda_{mn} \in M(\Lambda) \mid \varepsilon(\lambda_{mn}) = \varepsilon^0\}$.

Proposition 2.3. All possible neighbor distances in the set $M(p_j) = \{\lambda_{mn} = m + \alpha n \mid m, n \geqslant 0;\ 0 \leqslant \lambda_{mn} < p_j\}$ can be only elements of the set
$E^{(j)} = \{\varepsilon_{j-1}\} \cup E_j \cup E_{j-1} \cup \cdots \cup E_1 = \{\varepsilon_{j-1};\ \varepsilon_i + k\varepsilon_{i-1},\ 1 \leqslant k \leqslant a_i,\ i \leqslant j\}$
and for $j \to \infty$

$$\Pr A(\varepsilon_{j-1}; p_j) = (1 - \beta_j)^2 + O(2^{-j/2})$$

$$\Pr A(\varepsilon_j + \varepsilon_{j-1}; p_j) = \beta_j^2 + O(2^{-j/2})$$

$$\Pr A(\varepsilon_j + k\varepsilon_{j-1}; p_j) = 2\beta_j^2 + O(2^{-j/2}), \qquad 2 \leqslant k \leqslant a_j$$

$$\Pr A(\varepsilon_i + k\varepsilon_{i-1}; p_j) = 2\beta_j^2 \cdots \beta_i^2 + O(2^{-j/2}), \qquad 1 \leqslant k \leqslant a_j, \quad i < j$$

where $\beta_i = [a_i, a_{i-1}, ..., a_1]$.

Remark. Here and below $O(2^{-j/2})$ means a remainder which is estimated by $C2^{-j/2}$ with some absolute constant C.

Proof. Denote $l_j = [p_j/\alpha] + 1 = q_j - [(-1)^j \varepsilon_j/\alpha]$, so that $l_j = q_j$, if j is even and $l_j = q_j + 1$ if j is odd. Consider first the case when j is even. Let $\tilde{M}(p_j) = M(p_j) \cup \{p_j\} = \{\lambda_{mn} = m + \alpha\ n \mid m, n \geqslant 0;\ 0 \leqslant \lambda_{mn} \leqslant p_j\}$. Then $\tilde{M}(p_j) = \bigcup_{k=1}^{p_j} M_k$, so by Proposition 2.1 the set of neighbour distances in $\tilde{M}(p_j)$ coincides with the same in all the sets N_l with $l \leqslant l_j = q_j$. By Proposition 2.2 and its Corollary this set is $E^{(j)}$. Since $M(p_j) = \tilde{M}(p_j) \setminus \{p_j\}$ the same is valid for $M(p_j)$. Next, by formulas (2.1) (see also Fig. 1)

$$\Pr A(\varepsilon_{j-1}; p_j) = \frac{\sum_{0 < l \leqslant q_j} \lambda(l; \varepsilon_{j-1})}{\sum_{0 < l \leqslant q_j} l} = \frac{\sum_{q_{j-1} < l \leqslant q_j} (l - q_{j-1})}{\sum_{0 < l \leqslant q_j} l}$$

$$= \frac{(q_j - q_{j-1})^2}{q_j^2} + O\left(\frac{1}{q_j}\right)$$

Recall that $q_{j-1}/q_j = \beta_j$. Besides, $q_j = a_j q_{j-1} + q_{j-2} = (a_j a_{j-1} + 1) q_{j-2} + a_j q_{j-3} \geqslant 2q_{j-2}$, so $q_j > C^{-1} 2^{j/2}$; hence

$$\frac{1}{q_j} < C \cdot 2^{-j/2} \tag{2.3}$$

It gives that $\Pr A(\varepsilon_{j-1}; p_j) = (1 - \beta_j)^2 + O(2^{-j/2})$, which was stated. Next,

$$\Pr A(\varepsilon_{j-1} + \varepsilon_j; p_j) = \frac{\sum_{0 < l \leqslant q_j} \lambda(l; \varepsilon_{j-1} + \varepsilon_j)}{\sum_{0 < l \leqslant q_j} l} = \frac{\sum_{q_j - q_{j-1} < l \leqslant q_j} (l - q_j + q_{j-1})}{\sum_{0 < l \leqslant q_j} l}$$

$$= \frac{q_{j-1}^2}{q_j^2} + O\left(\frac{1}{q_j}\right) = \beta_j^2 + O(2^{-j/2})$$

which was stated. Similarly, one can calculate $\Pr A(\varepsilon_j + k\varepsilon_{j-1}; p_j)$, $2 \leqslant k \leqslant a_j$. Now, by the Corollary of Proposition 2.2,

$$\Pr A(\varepsilon_i + k\varepsilon_{i-1}; p_j) = \frac{\sum_{0<l\leqslant q_j} \lambda(l; \varepsilon_i + k\varepsilon_{i-1})}{\sum_{0<l\leqslant q_j} l}$$

$$= 2\frac{q_{i-1}^2}{q_j^2} + O\left(\frac{1}{q_j}\right) = 2\beta_j^2 \cdots \beta_i^2 + O(2^{-j/2})$$

For even j, the Proposition is proved. When j is odd, the set of neighbor distances in $\tilde{M}(p_j)$ coincides with the same in all the sets N_l, $l \leqslant l_j = q_j + 1$, so in comparison with the case of even j we have an additional set N_{q_j+1}. Since $|N_{q_j+1}| = q_j + 1$ all the probabilities $\Pr A(\varepsilon_i + k\varepsilon_{i-1}; p_j)$ change in $O(1/q_j)$, which is by, (2.3), $O(2^{-j/2})$. This remark proves Proposition 2.3 completely.

Introduce the function $s = s(\lambda_{mn})$ of *normalized* neighbor distances in $M(p_j)$,

$$s = \frac{\varepsilon}{\varepsilon_{j-1}} = \frac{\varepsilon(\lambda_{mn})}{\varepsilon_{j-1}}$$

Proposition 2.4. All possible normalized neighbor distances in $M(p_j)$ can be only elements of the set

$$S^{(j)} = \{1\} \cup S_j \dot{\cup} S_{j-1} \cup \cdots \cup S_1$$

where

$$S_j = \{k + r_j,\ 1 \leqslant k \leqslant a_j\}, \qquad S_i = \left\{\frac{1}{r_{j-1} \cdots r_i}(k + r_i),\ 1 \leqslant k \leqslant a_i\right\}, \qquad 1 \leqslant i < j$$

and

$$\Pr\{s = 1\} = (1 - \beta_j)^2 + O(2^{-j/2)}$$

$$\Pr\{s = 1 + r_j\} = \beta_j^2 + O(2^{-j/2})$$

$$\Pr\{s = k + r_j\} = 2\beta_j^2 + O(2^{-j/2}), \qquad 2 \leqslant k \leqslant a_j$$

$$\Pr\left\{s = \frac{1}{r_{j-1} \cdots r_i}(k + r_i)\right\} = 2\beta_j^2 \cdots \beta_i^2 + O(2^{-j/2}), \qquad 1 \leqslant k \leqslant a_i$$

Proof. All possible values of $s = \varepsilon/\varepsilon_{j-1}$ are

$$\frac{\varepsilon_i + k\varepsilon_{i-1}}{\varepsilon_{j-1}} = \frac{\varepsilon_{i-1}}{\varepsilon_{j-1}}\left(k + \frac{\varepsilon_i}{\varepsilon_{i-1}}\right) = \frac{1}{r_{j-1} \cdots r_i}(k + r_i)$$

which was stated.

3. CONSTRUCTION OF DUAL GAUSS DISTRIBUTION

This section is auxiliary. Here we prove the existence of $\lim_{n\to\infty} \beta_n = \beta$, where $\beta_n = [a_n, a_{n-1},..., a_1]$ and $\alpha = [a_1, a_2,...]$ obeys the Gauss distribution $d\alpha/[(1+\alpha)\ln 2]$. We call the distribution of β the dual Gauss distribution and we study some properties of it.

Proposition 3.1. Let $\alpha = [a_1, a_2,...]$ obey the Gauss distribution $d\alpha/[(1+\alpha)\ln 2]$. Then there exists a limit of $\beta_n = [a_n, a_{n-1},..., a_1]$, when $n \to \infty$, $\beta = w\text{-}\lim_{j\to\infty} \beta_n$.

Proof. Consider for $N > n$ the distribution ν_{Nn} of $\beta_{Nn} \equiv [a_N, a_{N-1},..., a_{N-n+1}]$. Since $\beta_{Nn}(\alpha) = \beta_{N-1,n}(G[\alpha])$, the distribution ν_{Nn} does not depend on N, $\nu_{Nn} = \nu_n$. Besides, ν_n are evidently compatible in the sense that

$$\nu_{n-1}[a_N, a_{N-1},..., a_{N-n+2}] = \sum_{a_{N-n+1} \in \mathbf{N}} \nu_n[a_N, a_{N-1},..., a_{N-n+1}]$$

The celebrated Kolmogorov theorem states that there exists $\lim_{n\to\infty} \nu_n = \nu$, and ν_n are the finite-dimensional distribution of ν. It means that there exists a weak limit β of β_n, when $n \to \infty$, and μ is the distribution of β. Proposition 3.1 is proved.

Let $b_1,..., b_n \in \mathbf{N}$. Denote $V[b_1,..., b_n] = \{\alpha = [a_1, a_2,...] \mid a_1 = b_1,..., a_n = b_n\}$, which is the segment between the points $[b_1,..., b_n]$ and $[b_1,..., b_n + 1]$. If

$$\frac{p_n}{q_n} = [b_1,..., b_n]$$

$$\frac{p'_n}{q'_n} = [b_1,..., b_n + 1]$$

then

$$p'_n = (b_n + 1) p_{n-1} + p_{n-2} = p_n + p_{n-1}$$

$$q'_n = (b_n + 1) q_{n-1} + q_{n-2} = q_n + q_{n-1}$$

so

$$\left| \frac{p_n}{q_n} - \frac{p'_n}{q'_n} \right| = \frac{|p_n q_{n-1} - q_n p_{n-1}|}{q_n q'_n} = \frac{1}{q_n q'_n} < 2^{2-n}$$

It means that the Lebesque measure of $V[b_1,..., b_n]$ is estimated as

$$|V[b_1,..., b_n]| < 2^{2-n}$$

Thus

$$[0, 1] = \bigcup_{b_1,..., b_n \in \mathbf{N}} V[b_1,..., b_n]$$

is a partition of the segment [0, 1] into small segments of length less than 2^{2-n}. Since the Gauss density is bounded, a similar estimate is valid for the invariant measure:

$$\mu(V[b_1,..., b_n]) \equiv \int_{V[b_1,..., b_n]} \frac{d\alpha}{(1+\alpha)\ln 2} < C \cdot 2^{-n} \tag{3.1}$$

It is noteworthy that if μ is the Gauss distribution and ν is the distribution of $\beta = \lim_{n \to \infty} \beta_n$, then for any segment $V[b_1,..., b_n] = [[b_1,..., b_n], [b_1,..., b_n + 1]]$,

$$\nu(V[b_1,..., b_n]) = \mu(V[b_n,..., b_1]) \tag{3.2}$$

It enables one to construct the distribution of β in the following way. Consider the Gauss distribution $\mu(d\alpha) = d\alpha/[(1+\alpha)\ln 2]$ as a measure on the set of half-infinite sequences $(a_1, a_2,...)$, $a_j \in \mathbb{N}$, where $\alpha = [a_1, a_2,...]$. Then the Gauss map G: $\alpha \to \{1/\alpha\}$ is the shift $(a_1, a_2,...) \to (a_2, a_3,...)$, so the G-invariance of μ gives that for any $n \geqslant 1$,

$$\mu(a_1, a_2,..., a_n) = \sum_{a=1}^{\infty} \mu(a, a_1, a_2,..., a_n)$$

Consider for any $N \geqslant 1$ a measure μ_N on the set of half-infinite sequences

$$(a_{-N+1}, a_{-N+2},...), \qquad a_j \in \mathbf{N}$$

which is the shift of μ in N steps to the left, so that

$$\Pr_{\mu_N}\{a_{-N+1} = b_1,..., a_{-N+n} = b_n\} = \Pr_{\mu}\{a_1 = b_1,..., a_n = b_n\}$$

for any $b_1,..., b_n \in \mathbf{N}$, $n \geqslant 1$. The G-invariance of μ implies that the measures μ_N are compatible in the sense that for $M > N > 0$,

$$\mu_M(a_{-N+1}, a_{-N+2},..., a_n) = \mu_N(a_{-N+1}, a_{-N+2},..., a_n)$$

Hence by the Kolmogorov theorem there exists a limit measure μ_∞ on the set of infinite sequence $\{a_j,, j \in \mathbf{Z}\}$ which is compatible with all μ_N. It is called the *natural extension* of μ. Let

$$R: \quad \{a_j, j \in \mathbf{Z}\} \to \{a_{-j}, j \in \mathbf{Z}\}$$

be the reflection and $R^*\mu_\infty$ be the distribution of $\{a_{-j},\ j\in \mathbf{Z}\}$. Consider the distribution $R^*\mu_\infty(a_1, a_2,...)$ as a measure $\nu(d\beta)$ on $[0, 1]$, where $\beta = [a_1, a_2,...]$. Then $\nu(d\beta)$ is just the distribution of β.

Proposition 3.2. The distribution $\nu(d\beta)$ of $\beta = \lim_{n\to\infty} \beta_n$, constructed in Proposition 3.1, has the following properties:

(i) It is invariant with respect to the Gauss map G.

(ii) It has no atoms.

(iii) It is singular with respect to the Lebesgue measure.

(iv) The support of $\nu(d\beta)$ coincides with $[0, 1]$.

Proof. We have $\sum_{b_1=1}^{\infty} \mu(V[b_n,..., b_1]) = \mu(V[b_n,..., b_2])$, which is just the compatibility condition for μ, so

$$\nu(V[b_2,..., b_n]) = \mu(V[b_n,..., b_2]) = \sum_{b_1=1}^{\infty} \mu(V[b_n,..., b_2, b_1])$$

$$= \sum_{b_1=1}^{\infty} \nu(V[b_1, b_2,..., b_n])$$

which means the invariance of ν with respect to the shift $(b_1, b_2,...) \to (b_2, b_3,...)$, or the G-invariance of $\nu(d\beta)$. Next, the estimate

$$\nu(V[b_1,..., b_n]) = \mu(V[b_n,..., b_1]) < C\cdot 2^{-n} \tag{3.3}$$

which follows from (3.1) and (3.2), implies that ν has no atoms. A direct calculation shows that

$$\nu(V[b_1, b_2]) = \mu(V[b_1, b_1]) = \frac{1}{\ln 2}\left|\ln\frac{b_1}{b_1b_2+1} - \ln\frac{b_1+1}{b_1b_2+b_2+1}\right|$$

$$\not\equiv \mu(V[b_1, b_2])$$

so $\nu \neq \mu$. Remark that μ is ergodic with respect to G (see ref. 6); hence μ_∞ is ergodic with respect to shifts and so ν is ergodic with respect to G (see again ref. 6). Since μ is absolutely continuous, it implies that ν is singular. Finally, (3.2) implies that

$$\nu(V[b_1,...,b_n]) > 0$$

for any $V[b_1,..., b_n]$, so the support of ν coincides with $[0, 1]$. Proposition 3.2 is proved.

The natural extension μ_∞ can be realized as a probability measure on the unit square $I^2 = [0, 1] \times [0, 1]$. Denote by Ω the set of sequences

$\{a_j \in \mathbf{N},\ j \in \mathbf{Z}\}$, so that μ_∞ is a probability measure on Ω. Define the map $\tau: \Omega \to I^2$ as

$$\tau: \quad \{a_j, j \in \mathbf{Z}\} \to (\alpha, \beta) = ([a_1, a_2, ...], [a_0, a_{-1}, a_{-2}, ...])$$

One can see easily that τ is a one-to-one measurable map from Ω to $I_{\text{irr}} \times I_{\text{irr}}$, where $I_{\text{irr}} = I \backslash \mathbf{Q}$ is the set of irrational numbers in $[0, 1]$. It enables one to define the probability measure $\tau^* \mu_\infty$ which can be viewed as the realization of μ_∞ on I^2. For the sake of brevity we shall denote $\tau^* \mu_\infty$ by μ_∞. Note that for any $N_1, N_2 > 0$,

$$\mu_\infty(V[b_1, ..., b_{N_2}] \times V[b_0, b_{-1}, ..., b_{-N_1+1}]) = \mu(V[b_{-N_1+1}, ..., b_{N_2}]) \tag{3.4}$$

Proposition 3.3. The probability measure $\mu_\infty(d\alpha\, d\beta)$ on I^2 has the following properties:

(i) It is invariant and ergodic with respect to the map

$$G_\infty: \quad (\alpha, \beta) \to \left(\left\{ \frac{1}{\alpha} \right\}, \frac{1}{[1/\alpha] + \beta} \right) \tag{3.5}$$

(ii) It has no atoms.

(iii) It is singular with respect to the Lebesgue measure $d\alpha\, d\beta$.

(iv) The support of $\mu_\infty(d\alpha\, d\beta)$ coincides with I^2.

(v)

$$\int_{\{\beta \in I\}} \mu_\infty(d\alpha\, d\beta) = \mu(d\alpha); \qquad \int_{\{\alpha \in I\}} \mu_\infty(d\alpha\, d\beta) = \nu(d\beta)$$

Proof. The map G_∞ corresponds to the shift $\{a_j,\ j \in \mathbf{Z}\} \to \{a_{j+1}, j \in \mathbf{Z}\}$, so the G_∞-invariance and the ergodicity of μ_∞ follow from the G-invariance and the ergodicity of the Gauss measure $\mu(d\alpha)$. All the other statements follow from (3.4) and Proposition 3.2. Proposition 3.3 is proved.

Let

$$\begin{aligned} S: &\quad (\alpha, \beta) \to (\beta, \alpha) \\ \pi_1: &\quad (\alpha, \beta) \to \alpha, \qquad i_1: \quad \alpha \to (\alpha, 0) \\ \pi_2: &\quad (\alpha, \beta) \to \beta, \qquad i_2: \quad \beta \to (0, \beta) \end{aligned}$$

Then the map G_∞ satisfies the relations

$$G_\infty S G_\infty S = \mathrm{Id} \tag{3.6}$$

which is the identity map, and

$$\pi_1 G_\infty^n i_1 = G^n, \qquad n \geqslant 0 \tag{3.7}$$

(3.6) means that S is the *symmetry transformation* for G_∞, so that G_∞ is *invertible* and

$$G_\infty^{-1} = S G_\infty S \tag{3.8}$$

If $\rho(d\alpha\, d\beta) = p(\alpha, \beta)\, d\alpha\, d\beta$ is an absolutely continuous probability measure on I^2, define $G_\infty^* \rho(d\alpha\, d\beta)$ as

$$\int_A G_\infty^* \rho(d\alpha\, d\beta) = \int_{G_\infty^{-1}(A)} \rho(d\alpha\, d\beta)$$

for any Borel set $A \subset I^2$.

Proposition 3.4. Let $p(\alpha)\, d\alpha$ be an absolutely continuous probability measure on $I = [0, 1]$. Then

$$w\text{-}\lim_{n\to\infty} (G_\infty^*)^n p(\alpha)\, d\alpha\, d\beta = \mu_\infty(d\alpha\, d\beta)$$

Proof. We have

$$\iint_{V[b_1,\dots,b_{N_2}] \times V[b_0, b_{-1}, \dots, b_{-N_1+1}]} (G_\infty^*)^n p(\alpha)\, d\alpha\, d\beta$$

$$= \iint_{V[b_{-N_1+1},\dots,b_{N_2}] \times I} (G_\infty^*)^{n-N_1} p(\alpha)\, d\alpha\, d\beta$$

$$= \int_{V[b_{-N_1+1},\dots,b_{N_2}]} (G^*)^{n=-N_1} p(\alpha)\, d\alpha$$

The Gauss map G is mixing (see ref. 6), so

$$\lim_{n\to\infty} \int_{V[b_{-N_1+1},\dots,b_{N_2}]} (G^*)^{n-N_1} p(\alpha)\, d\alpha$$

$$= \int_{V[b_{-N_1+1},\dots,b_{N_2}]} \mu(d\alpha)$$

$$= \iint_{V[b_1,\dots,b_{N_2}] \times V[b_0, b_{-1},\dots,b_{-N_1+1}]} \mu_\infty(d\alpha\, d\beta)$$

Hence,

$$\lim_{n\to\infty} \iint_A (G_\infty^*)^n p(\alpha)\, d\alpha\, d\beta = \iint_A \mu_\infty(d\alpha\, d\beta)$$

for any $A = V[b_1,..., b_{N_2}] \times V[b_0, b_{-1},..., b_{-N_1+1}]$, which implies the weak convergence of $(G^*_\infty)^n p(\alpha)\, d\alpha\, d\beta$ to $\mu_\infty(d\alpha\, d\beta)$. Proposition 3.4 is proved.

Corollary of Proposition 3.4. Let $\alpha = [a_1, a_2,...]$ be a random variable with an absolutely continuous distribution $p(\alpha)\, d\alpha$ on $[0, 1]$ and $r_n = [a_{n+1}, a_{n+2},...]$, $\beta_n = [a_n, a_{n-1},..., a_1]$, $\gamma_n = (r_n, \beta_n) \in I^2$. Then

$$w\text{-}\lim_{n \to \infty} \gamma_n = \gamma$$

where γ is μ_∞-distributed.

Proof. It is simply a reformulation of Proposition 3.4.

Remark finally that the dual measure can be constructed for any G-invariant measure μ.

4. LIMIT SPACING DISTRIBUTION

In this section we apply the formulas of Section 2 to deduce Theorems 1.1–1.5. We shall assume that α is a random variable with an absolutely continuous distribution $p(\alpha)\, d\alpha$ on $[0, 1]$. According to Proposition 2.4, the normalized spacing s in $M(p_j)$ takes values in the set $S^{(j)} = \{1\} \cup S_j \cup S_{j-1} \cup \cdots \cup S_1 \equiv \{1 = s_{j0} < s_{j1} < s_{j2} < \cdots\}$.

Proposition 4.1. There exists $w\text{-}\lim_{j \to \infty} s_{jl} = s_l$ and the distribution of s_l does not depend on $p(\alpha)\, d\alpha$.

Proof. Consider some s_{jl}. We shall assume that l is fixed and $j > l$. Let $s_{jl} \in S_m$. Then

$$l = a_j + \cdots + a_{m+1} + k \tag{4.1}$$

where $1 \leqslant k \leqslant a_m$, and by Proposition 2.4

$$s_{jl} = \frac{1}{r_{j-1} \cdots r_m} (k + r_m) \tag{4.2}$$

Note that (4.1) implies that $m \geqslant j - l$, so by (4.1) and (4.2), s_{jl} is determined uniquely by $a_j,..., a_{j-l+1}$ and $r_j,..., r_{j-l}$. Moreover, since

$$r_{j-n} = G^{l-n}[r_{j-l}], \qquad a_{j-n} = AG^{l-n+1}[r_{j-l}]$$

s_{jl} is a function of only r_{j-l}:

$$s_{jl} = H_l(r_{j-l}) \tag{4.3}$$

where the function H_l does not depend on j.

Show that H_l is a piecewise fractional linear function and it is nonconstant on any segment. To this end, put $j = l$. Then (4.3) reads

$$s_{ll} = H_l(\alpha) \tag{4.4}$$

since $r_0 = \alpha$. Now, by Proposition 2.3,

$$s_{ll} = \frac{\varepsilon_i + k\varepsilon_{i-1}}{\varepsilon_l} \tag{4.5}$$

with some $i \leqslant l+1$ and $1 \leqslant k \leqslant a_i$. Since $\varepsilon_i = (-1)^{i-l}(q_i\alpha - p_i)$, we get from (4.4), (4.5) that

$$H_l(\alpha) = (-1)^{i-l} \frac{(q_i - kq_{i-1})\alpha - (p_i - kp_{i-1})}{q_l\alpha - p_l}$$

$$= (-1)^{i-l} \frac{(q_{i-2} + tq_{i-1})\alpha - (p_{i-2} + tp_{i-1})}{q_l\alpha - p_l}$$

where $t = a_i - k$. It proves that H_l is a fractional linear function of α on any segment $V[b_1,..., b_l]$. The last formula can be rewritten as

$$H_l(\alpha) = C \frac{\alpha - (p_{i-2} + tp_{i-1})/(q_{i-2} + tq_{i-1})}{\alpha - p_l/q_l}$$

where C does not depend on α. Since

$$\frac{p_{i-2} + tp_{i-1}}{q_{i-2} + tq_{i-1}} \neq \frac{p_l}{q_l}$$

(because p_n/q_n converges monotonously to α for even and odd n's), we get that $H_l(\alpha)$ is nonconstant on any segment.

Return now to Eq. (4.3). By Proposition 3.3, $r_{j-l} = G^{j-l}[\alpha] \to r$ when $j \to \infty$, where r has the distribution $dr/[(1+r)\ln 2]$. Since H_l is a bounded fractionally linear function on any segment $V[b_1, ..., b_l]$, it implies that

$$w\text{-}\lim_{j\to\infty} s_{jl} = w\text{-}\lim_{j\to\infty} H_l(r_{j-l}) = H_l(r) \tag{4.6}$$

Proposition 4.1 is proved.

Proposition 4.1 ensures the convergence of s_{jl} when $j \to \infty$. Now we show the convergence of the probabilities $\pi_{jl} \equiv \Pr\{s = s_{sl}\}$ to a limit when $j \to \infty$.

Proposition 4.2. For any fixed $l = 0, 1, 2,...$ there exists $w\text{-}\lim_{j\to\infty} \pi_{jl} = \pi_l$ and the distribution of π_l does not depend on $p(\alpha)\, d\alpha$.

Proof. Let for definiteness $l \geqslant 2$ and $s_{jl} \in S_m$, so that (4.1) holds and by Proposition 2.4

$$\pi_{jl} = 2\beta_j^2 \cdots \beta_m^2 + O(2^{-j/2})$$

where the remainder term is uniform in α and l. We should prove the existence of $\lim_{j \to \infty} \hat{\pi}_{jl} = \pi_l$, where

$$\hat{\pi}_{jl} = 2\beta_j^2 \cdots \beta_m^2 \tag{4.7}$$

By (4.1), $m \geqslant j - l$, so $\hat{\pi}_{jl}$ is determined uniquely by $a_j, \ldots, a_{j-l+1}$ and $\beta_j, \ldots, \beta_{j-l}$. Moreover, since $\beta_n = [a_n, a_{n-1}, \ldots, a_1] = G^{j-n}[\beta_j]$, $n \leqslant j$, and $a_n = AG^{j-n+1}[\beta_j]$, $\hat{\pi}_{jl}$ is a function of β_j,

$$\hat{\pi}_{jl} = R_l(\beta_j)$$

In any segment $\beta_l \in V[b_1, \ldots, b_l]$,

$$\begin{aligned}\hat{\pi}_{ll} = R_l(\beta_l) = 2\beta_l^2 \cdots \beta_m^2 &= 2\beta_l^2 (G[\beta_l])^2 \cdots (G^{l-m}[\beta_l])^2 \\ &= 2\varepsilon_{l-m}^2(\beta_l) = 2\,|q_{l-m}\beta_l - p_{l-m}|^2\end{aligned}$$

Hence $R_l(\beta)$ is a quadratic nonconstant function of β in any segment $V[b_1, \ldots, b_l]$.

By Proposition 3.3, $w\text{-}\lim_{j \to \infty} \beta_j = \beta$, where β obeys the dual Gauss distribution, so

$$\underset{j \to \infty}{w\text{-}\lim}\ \pi_{jl} = \underset{j \to \infty}{w\text{-}\lim}\ \hat{\pi}_{jl} = \underset{j \to \infty}{w\text{-}\lim}\ R_l(\beta_j) = R_l(\beta) \tag{4.8}$$

Proposition 4.2 is proved.

Since Propositions 4.1 and 4.2 give the convergence of s_{jl}, π_{jl} only for fixed l, it is useful to have a uniform bounds for s_{jl}, π_{jl}.

Proposition 4.3. An absolute constant C exists such that for any j, l

$$2^l \geqslant s_{jl} \geqslant \frac{l}{2} \qquad \text{and} \qquad \pi_{jl} < \frac{C}{l^2}$$

Proof. Prove first that $2^l \geqslant s_{jl}$. For $l = 0$: $2^l = 1 = s_{jl}$, so it is valid. Assume that $2^l \geqslant s_{jl}$ and prove that $2^{l+1} \geqslant s_{j,l+1}$. Two cases are possible: either s_{jl}, $s_{j,l+1}$ belong to the same series

$$S_m = \left\{ \frac{1}{r_{j-1} \cdots r_m} (k + r_m),\ 1 \leqslant k \leqslant a_m \right\}$$

or they belong to different series. In both cases, however,

$$\frac{s_{j,l+1}}{s_{jl}}=\frac{k+1+r_m}{k+r_m}\leqslant\frac{k+1}{k}\leqslant 2$$

so $s_{j,l+1}\leqslant 2^{l+1}$, which was stated.

Prove now that $s_{jl}\geqslant l/2$. We have

$$s_{j0}=1>0$$

$$s_{j2}>s_{j1}=1+r_j>1$$

so for $l=0, 1, 2$ the estimate $s_{jl}\geqslant l/2$ is valid. Assume that

$$s_{j,l-2}\geqslant(l-2)/2 \qquad \text{for some} \quad l\geqslant 3$$

and prove thnat $s_{jl}\geqslant l/2$. Two cases are possible: either $s_{j,l-2}$, $s_{j,l-1}$, s_{jl} belong to different series

$$S_m=\left\{\frac{1}{r_{j-1}\cdots r_m}(k+r_m),\ 1\leqslant k\leqslant a_m\right\}$$

or at least two of them belong to the same series S_m. In the second case

$$s_{jl}-s_{j,l-2}\geqslant\frac{1}{r_{j-1}\cdots r_m}>1$$

which implies that

$$s_{jl}\geqslant\frac{l-2}{2}+1=\frac{l}{2}$$

In the first case

$$s_{j,l-2}=\frac{1}{r_{j-1}\cdots r_m}(a_m+r_m)=\frac{1}{r_{j-1}\cdots r_{m-1}}$$

$$s_{j,l-1}=\frac{1}{r_{j-1}\cdots r_{m-1}}(1+r_{m-1})$$

$$s_{jl}=\frac{1}{r_{j-1}\cdots r_{m-2}}(1+r_{m-2})$$

so

$$\frac{s_{jl}}{s_{j,l-2}}=\frac{1+r_{m-2}}{r_{m-2}}\geqslant 2$$

Hence

$$s_{jl} \geqslant 2s_{j,l-2} \geqslant l-2 \geqslant l/2 \qquad \text{if} \quad l \geqslant 4$$
$$s_{j3} \geqslant 2s_{j1} \geqslant 2 > 3/2$$

which was stated.

Estimate now π_{jl}. Let $k_j = [p_j/\alpha] = q_j$ or $q_j - 1$. If ε_{jl} belongs to the mth series E_m, i.e., if

$$\varepsilon_{jl} = \varepsilon_m + i\varepsilon_{m-1}$$

then by (2.2) (with k, m, i instead of l, j, k, respectively, in that formula)

$$\pi_{jl} = \frac{\sum_{0 \leqslant k < k_j} \lambda(k; \varepsilon_{lj})}{\sum_{0 \leqslant k \leqslant k_j} k} = 2\frac{q_{m-1}^2}{k_j^2 - k_j} = 2\frac{q_{m-1}^2}{k_j^2 - k_j}$$
$$= 2\frac{q_{m-1}^2}{q_j^2}\left[1 + O\left(\frac{1}{q_j}\right)\right] \tag{4.9}$$

Note that

$$l \leqslant a_j + \cdots + a_m \tag{4.10}$$

On the other hand

$$\frac{q_j}{q_{m-1}} \geqslant \frac{a_j + \cdots + a_m}{2} \tag{4.11}$$

Indeed,

$$q_j \geqslant (a_j a_{j-1} + 1) q_{j-2} \geqslant (a_j + a_{j-1}) q_{j-2} \geqslant (a_j + a_{j-1})(a_{j-2} + a_{j-3}) q_{j-4}$$
$$\geqslant (a_j + a_{j-1} + a_{j-2} + a_{j-3}) q_{j-4} \geqslant \cdots$$

and

$$q_j \geqslant a_j q_{l-1} \geqslant \frac{a_j + 1}{2} q_{j-1} \geqslant \frac{a_j + 1}{2} \quad (a_{j-1} + a_{j-2}) q_{j-3}$$
$$\geqslant \frac{a_j + a_{j-1} + a_{j-2}}{2} q_{j-3} \geqslant \cdots$$

which implies (4.11). By (4.9)–(4.11) we get that

$$\pi_{jl} \leqslant \frac{C}{l^2}$$

which was stated. Proposition 4.3 is proved.

It is noteworthy that the estimates of Proposition 4.3 cannot be improved essentially. Namely, if a_j is large and $l = a_j$, then

$$s_{jl} = l + r_j < l + 1$$

$$\pi_{jl} = 2\frac{q_{j-1}^2}{q_j^2} \sim \frac{2}{l^2}$$

We prove now Theorems 1.1–1.5.

Proof of Theorem 1.1. Follows from Propositions 4.1, 4.2.

Proof of Theorem 1.2. Follows from Proposition 4.3.

Proof of Theorem 1.3. We have by (4.3) that

$$s_{jl} = H_l(r_{j-l}) = H_l(P_{-l}G_\infty^j \eta)$$

where $\eta = (a_k,\ k = 1, 2,...)$, G_∞^j: $(a_k,\ k = 1, 2,...) \to (a'_k = a_{k+j},\ k = -j+1, -j+2,...)$ and P_{-l}: $(a_k, k = -j+1,\ -j+2,...) \to [a_{-l+1}, a_{-l+2},...]$, $j \geq l$. By the Corollary of Proposition 3.4

$$\underset{j\to\infty}{w\text{-lim}}\ P_{-l}G_\infty^j \eta = P_{-l}\gamma$$

so

$$\underset{j\to\infty}{w\text{-lim}}\ s_{jl} = H_l(P_{-l}\gamma) \equiv F_l(\gamma)$$

In addition, by (4.8)

$$\underset{j\to\infty}{w\text{-lim}}\ \pi_{jl} = \underset{j\to\infty}{w\text{-lim}}\ R_l(\beta_j) = \underset{j\to\infty}{w\text{-lim}}\ R_l(Q_j G_\infty^j \eta) = R_l(Q(\gamma))$$

where Q_j: $(a_k, k = -j+1, -j+2,...) \to [a_0, a_{-1},..., a_{-j+1}]$ and Q: $(a_k, k \in \mathbf{Z}) \to [a_0, a_{-1}, a_{-2},...]$. Redenoting $R_l(Q\gamma)$ by $R_l(\gamma)$, we get that $\pi_l = R_l(\gamma)$. Theorem 1.3 is proved.

Proof of Theorem 1.4. Let $\chi(x) = 1$ if $x \geq 0$, and $= 0$ if $x < 0$. Notice that

$$P_j(x) = \int_{-\infty}^x \rho_j(ds) = \sum_{\{jl \mid s_{jl} \leq x\}} \pi_{jl} = \sum_{l=0}^\infty \chi(x - s_{jl})\pi_{jl}$$

where because of Theorem 1.2 the sum is actually finite. So by Theorem 1.3

$$\underset{j\to\infty}{w\text{-lim}}\ P_j(x) = \sum_{l=0}^\infty \chi(x - F_l(\gamma))\, R_l(\gamma)$$

which proves Theorem 1.4.

Proof of Theorem 1.5. Remark that by Proposition 2.4, $\pi_{j0} = (1-\beta_j^2) + O(2^{-j/2})$ and $\beta_j = [a_j, a_{j-1},..., a_1]$ lies between $1/a_j$ and $1/(a_j+1)$. It is well known that for generic α, a_j strongly fluctuates when $j \to \infty$ and $\lim_{j\to\infty}(a_1 \cdots a_j)^{1/j}$ exists and it is finite, so β_j and hence π_{j0} have no limit when $j \to \infty$. Theorem 1.5 is proved.

5. DISCUSSION

In the present paper we have studied the limit distribution of the energy level spacing for the system of two harmonic oscillators with generic ratio of frequencies. The problem is reduced to a similar one for the set of levels $\{\lambda_{mn} = m + n\alpha \mid m, n \geqslant 0\}$, $\alpha = \omega_2/\omega_1$, and follows ref. 3 we consider the spacing distribution $\rho_j(ds)$ in the energy intervals $\{0 \leqslant \lambda_{mn} < \rho_j\}$, where $p_j/q_j = [a_1,..., a_j]$ are the approximants of $\alpha = [a_1, a_2,...]$. We have found the following properties of $\rho_j(ds)$:

(i) Discreteness.

(ii) Highly irregular behavior of $\rho_j(ds)$ when $j \to \infty$ for any fixed generic α.

(iii) Existence of $\lim_{j\to\infty} \rho_j(ds) = \rho(ds)$ for random α with any absolutely continuous distribution on $[0, 1]$.

(iv) Universality of the random limit distribution.

(v) Powerlike tail of $\rho(ds)$.

Discreteness of

$$\rho_j(ds) = \sum_l \pi_{jl}\, \delta(s - s_{jl})\, ds$$

means that each spacing s_{jl} has high multiplicity which is proportional to the whole number of levels so that the weight π_{jl} is of order of 1 for any fixed $l \geqslant 0$. Besides, for s_{jl} and π_{jl} we have uniform estimates $s_{jl} > l/2$ and $\pi_{jl} < C/l^2$. For any fixed α, s_{jl} and π_{jl} are determined by far coefficients a_n of the expansion of α in the continued fraction and so they behave very irregularly as $j \to \infty$.

To describe the behavior of $\rho_j(ds)$ for $j \to \infty$, introduce the space Ω of infinite sequences $\omega = \{a_n \in \mathbf{N},\ n \in \mathbf{Z}\}$. For any $\omega = (a_n \in \mathbf{N},\ n \in \mathbf{Z}\} \in \Omega$, define the following:

(i) The set

$$S(\omega) = \{1\} \cup S_0(\omega) \cup S_1(\omega) \cup S_2(\omega) \cup \cdots \equiv \{1 = s_0 < s_1(\omega) < s_2(\omega) < \cdots\}$$

where

$$S_0(\omega)=\{k+r_0,\ 1\leqslant k\leqslant a_0\}$$

$$S_i(\omega)=\left\{\frac{1}{r_{-1}\cdots r_{-i}}(k+r_{-i}),\ 1\leqslant k\leqslant a_{-i}\right\},\qquad i\geqslant 1$$

and

$$r_l\equiv[a_{l+1},a_{l+2},\dots]$$

(ii) The sequence $\{\pi_l(\omega),\ l\geqslant 0\}$ as

$$\pi_0(\omega)=(1-\beta_0)^2$$
$$\pi_1(\omega)=\beta_0^2$$
$$\pi_l(\omega)=2\beta_0^2, 1<l\leqslant a_0$$
$$\pi_l(\omega)=2\beta_0^2\cdots\beta_{-i}^2,\ a_0+\cdots+a_{-i+1}<l\leqslant a_0+\cdots+a_{-i},\ i\geqslant 1$$

where $\beta_l=[a_l,a_{l-1},a_{l-2},\dots]$. Put

$$\rho(ds;\omega)=\sum_{l\geqslant 0}\pi_l(\omega)\,\delta(s-s_l(\omega))\,ds \tag{5.1}$$

and

$$T_j:\quad \alpha=[a_1,a_2,\dots]\to\omega=\{\omega_n\in\mathbf{N},n\in\mathbf{Z}\}$$

with

$$\begin{aligned}\omega_n&=1 &&\text{if}\quad n\leqslant -j\\ &=a_{n+j} &&\text{if}\quad n>-j\\ T_j:&\quad [0,1]\to\Omega\end{aligned} \tag{5.2}$$

It is clear that T_j is the shift in j units to the left, continued by $\omega_n=1$ for $n\leqslant -j$. Then Proposition 2.4 implies the inequality

$$\sup_{x\in\mathbf{R}^1}\left|\int_{-\infty}^{x}\rho_j(ds)-\int_{-\infty}^{x}\rho(ds;T_j(\alpha))\right|\leqslant C2^{-j/2} \tag{5.3}$$

where C is an absolute constant. This means that there is a *family* of distributions (5.1), labeled by the parameter $\omega\in\Omega$, and for $j\to\infty$, $\rho_j(ds)$ is close to $\rho(ds;\omega)$ with $\omega=T_j(\alpha)$. It is worth noting that (5.3) remains valid for any continuation of ω_n for $n\leqslant -j$ in (5.2).

Generally, $T_j(\alpha)$ has no limit and its behavior is quite chaotic as $j \to \infty$. One can say that the system of two harmonic oscillators displays "quantum chaos" for generic α in the sense that the energy level spacing distribution $\rho_j(ds)$ shows the chaotic behavior as $j \to \infty$. This chaotic behavior is approximated according to (5.3) by the shifts

$$T^j: \quad \{a_n, n \in \mathbf{Z}\} \to \{a_{n+j}, n \in \mathbf{Z}\}$$

in the parameter space Ω.

If α is random on $[0, 1]$ with some absolutely continuous distribution $p(\alpha)\, d\alpha$, then $T_j(\alpha)$ weakly converges to the natural extension μ_∞ of the Gauss measure and the estimate (5.3) implies the convergence of $\rho_j(ds)$ when $j \to \infty$ to $\rho(ds; \omega)$, where ω is μ_∞-distributed.

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